A Unified Analysis of Supersonic Nonequilibrium Flow over a Wedge: I. Vibrational Nonequilibrium

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The paper begins with a survey and discussion of the existing analytical work on the problem of supersonic nonequilibrium flow over a wedge and certain related topics A new approach is formulated which provides a more complete picture of the flow field and retains greater generality The uniform freestream may be either in equilibrium or frozen in some out-of-equilibrium state The exact problem is simplified by assuming that the unknown quantities deviate only slightly from their frozen-flow values The resulting differential equations and boundary conditions are solved The physical situations in which this solution is a good first approximation are listed and explained In some of these situations, the solution can be further simplified to give an explicit picture of the flow field The simplified solution shows the existence of a relaxation region behind the shock wave and an entropy layer next to the wedge surface, in addition to the classical incoming and outgoing waves along which disturbances are propagated A numerical example, treated by using the simplified solution, is found to be in good qualitative agreement with existing numerical solution of the exact problem using the method of characteristics Causes of quantitative discrepancy are discussed, and possible improvements are suggested

Introduction

NONEQUILIBRIUM effects have become important because of the elevated temperatures encountered in high-speed flows, and much effort has accordingly been devoted to re-examining the classical problems of gasdynamics One problem of basic interest is supersonic flow over a wedge The new effects complicate the analysis by introducing a characteristic length, namely, a characteristic relaxation length of the gas, which destroys the self-similar nature of the classical flow field

Figure 1 gives a general picture of the new flow field The shock wave starts with the frozen-flow slope at the leading edge and curves to the equilibrium-flow slope The gas is, in general, out of equilibrium right behind the shock wave and relaxes in the relaxation zone as it flows downstream When it reaches the equilibrium core, the gas comes to equilibrium at a uniform state as given by the equilibrium oblique-shock relations The region next to the wedge surface is called the entropy layer, in which the flow comes to equilibrium far downstream at varying entropy, temperature, density, and speed, but constant pressure, from streamline to streamline These streamlines undergo different entropy changes as they pass through the shock wave at different angles and also as they relax at different rates in traversing the relaxation zone

Approximate solutions have been obtained by various authors in this and related problems by expanding the flow quantities about some reference state in terms of a small parameter or an independent variable The possible choice of reference state and small parameter is by no means unique, and each has its limitations in validity and results

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Linearized equations with the freestream as the reference state were originally used to study the case of a slender wedge The solution was achieved approximately by Moore and Gibson[1] and exactly by Clarke[2] Vincenti[3] generalized Clarke's result to the case where the freestream may be frozen out of equilibrium These works show how the disturbance created by the wedge is propagated (or, in other words, show the roles played by the frozen and equilibrium Mach lines) and how the various flow quantities vary on the wedge surface In spite of the restriction to a slender wedge, the linearized equations are still too complicated to be solved explicitly, so that an analytical picture of the flow field is yet to be seen Also, these works do not show a major new feature of nonequilibrium effects, namely, the entropy layer, which is of second-order magnitude in this perturbation scheme, although it can be computed numerically using first-order results

Using the facts that the flow is almost frozen near the leading edge and is almost in equilibrium far from the leading edge, Sedney[4] attempted to solve the exact problem by expanding the flow variables in powers of the dimensionless distance from the leading edge and its inverse, thus making perturbations about the known results of frozen and equilibrium flow, respectively Napolitano[5] attempted to solve in the same way the related problem of an expansion corner This approach eliminates the restriction to a slender wedge and describes certain nonlinear effects However, difficulties arise in seeking a uniformly valid first approximation in the equilibrium perturbation, which is of a singular nature, because of the presence of the relaxation layer and the entropy layer Although the relaxation layer can be treated by the same technique as is used for a viscous boundary layer, the details of the entropy layer are nearly impossible to determine because they depend on the history of the streamlines in the basically nonequilibrium region, which is as difficult to solve as the exact problem itself and lies outside the scope of this perturbation scheme Unless some global properties, such as additional conservation laws, can be invoked to help the determination of details of this layer, a uniformly valid first approximation is out of sight in the equilibrium end of this perturbation scheme

It is clearly desirable to have a scheme involving perturbations about either the frozen or the equilibrium flow but covering both limits in the results Spence,[6] in solving the analogous one-dimensional unsteady problem of a piston moved impulsively into a gas at rest, introduced such a scheme by assuming that the vibrational energy is never greatly excited in the vibrational nonequilibrium case, and that the gas is never greatly dissociated in the dissociation case, throughout the flow field Dressler[7] applied Spence's results to the wedge problem by invoking the equivalence principle of hypersonic small-disturbance theory However, he also failed to exhibit the entropy layer because it is lost in making the hypersonic small-disturbance approximation In the present analysis, Spence's assumption is shown to lead to a uniformly valid approximation containing all physical features for the wedge problem without any further approximation

The analysis that follows deals with vibrational nonequilibrium Dissociation will be dealt with in Part II These phenomena, although generally coupled in their effects, are usually analyzed independently of each other for convenience Their rate processes are of a different nature under present considerations; hence, each of them will be given a separate but parallel treatment, rather than be included in a general treatment as in Refs 1–3 The wedge under consideration is not necessarily slender and the freestream not necessarily in equilibrium The exact problem is attacked by perturbing about the frozen-flow solution without specifying a small parameter at the outset in order to preserve the generality A survey then is made of physical situations in which the solution to the first-order perturbation problem is a good first approximation to the solution of the exact problem For each of these situations, there is a small parameter corresponding to a particular perturbation scheme, to which results the solution of the general perturbation problem can be reduced accordingly The reductions carried out are the ones giving explicit results All of them are parameter perturbations, uniformly valid throughout the space behind the shock wave, except one, which is a coordinate perturbation valid in a certain limited region These results contain, in general, four terms that can be interpreted physically as the outgoing and incoming waves, the relaxation zone, and the entropy layer The results also show, as in Vincenti's slender-wedge case, that, when the freestream is properly out of equilibrium, the shock wave remains straight in spite of nonequilibrium effects, and the flow quantities stay constant behind the shock wave. Results for a 40° wedge with a freestream Mach number of 6 are compared with the numerical solution of the exact problem carried out by Sedney, South, and Geiber[8] using the method of characteristics

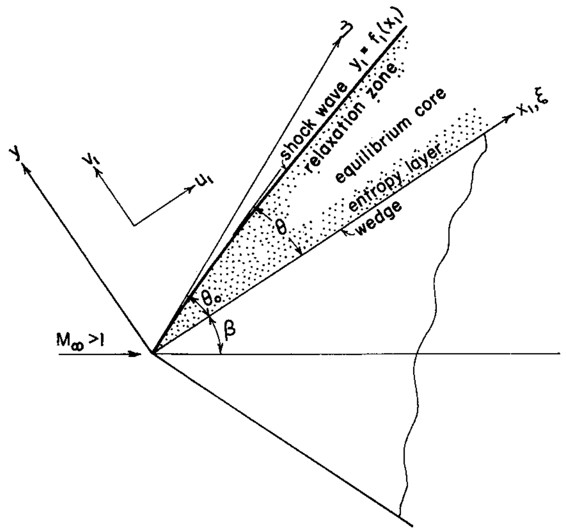

Fig 1 Supersonic nonequilibrium flow over wedge and its coordinate system

Exact Problem

The equations expressing conservation of mass, momentum, and energy for steady, two-dimensional flow of an inviscid, nonheat-conducting gas are

$$\frac{\partial}{\partial x_1}(\rho_1 u_1) + \frac{\partial}{\partial y_1}(\rho_1 v_1) = 0 \tag{1}$$

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial x_1} = 0 \tag{2}$$

$$u_1 \frac{\partial v_1}{\partial x_1} + v_1 \frac{\partial v_1}{\partial y_1} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial y_1} = 0 \tag{3}$$

$$u_1 \frac{\partial h_1}{\partial x_1} + v_1 \frac{\partial h_1}{\partial y_1} - \frac{1}{\rho_1}\left(u_1 \frac{\partial p_1}{\partial x_1} + v_1 \frac{\partial p_1}{\partial y_1}\right) = 0 \tag{4}$$

As a matter of later convenience, x_1 is measured along the wedge surface and y_1 perpendicular to it, and u_1 and v_1 are the velocities in the x_1 and y_1 directions, respectively; p_1 is the pressure, ρ_1 the density, and h_1 the enthalpy If the gas is capable of being excited in only one vibrational mode of internal energy, the thermal equation of state remains the same as that of a perfect gas, but the caloric equation of state differs by an additional term They are

$$p_1 = R\rho_1 T_1 \tag{5}$$

$$h_1 = [\gamma/(\gamma - 1)](p_1/\rho_1) + e_{v_1} \tag{6}$$

with e_{v_1}, a new variable, denoting the vibrational energy To complete the system a rate equation is needed and is taken in the form

$$u_1 \frac{\partial e_{v_1}}{\partial x_1} + v_1 \frac{\partial e_{v_1}}{\partial y_1} = \frac{e_{v_1}{}^* - e_{v_1}}{\tau_1} \tag{7}$$

Here $e_{v_1}{}^*$ is the fictitious value that e_{v_1} would assume if the gas were in equilibrium at the local temperature and is given as a function of temperature by

$$e_{v_1}{}^* = R\Theta/(e^{\Theta/T_1} - 1) \tag{8}$$

where the constant Θ is the characteristic temperature of molecular vibration Also in Eq (7), τ_1, a function of pressure and temperature in general, has the meaning of the relaxation time Its most commonly cited form is given by Landau and Teller[9] as follows:

$$\tau_1 = K_1 T_1^{1/6} e^{K_2/T_1^{1/3}} / p_1(1 - e^{-\Theta/T_1}) \tag{9}$$

where K_1 and K_2 are constants that depend on the physical properties of the gas molecule Only two thermodynamic variables besides e_{v_1} are needed to determine the thermodynamic state of the gas; therefore, if p_1 and ρ_1 are preferred, Eqs (4, 8, and 9) can be rewritten, by virtue of Eqs (5) and (6), as

$$u_1 \frac{\partial}{\partial x_1}\left(\frac{p_1}{\rho_1{}^\gamma}\right) + v_1 \frac{\partial}{\partial y_1}\left(\frac{p_1}{\rho_1{}^\gamma}\right) + \frac{(\gamma - 1)}{\rho_1{}^{\gamma - 1}}\left(u_1 \frac{\partial e_{v_1}}{\partial x_1} + v_1 \frac{\partial e_{v_1}}{\partial y_1}\right) = 0 \tag{10}$$

$$e_{v_1}{}^* = \frac{R\Theta}{e^{R\Theta\rho_1/p_1} - 1} \tag{11}$$

$$\tau_1 = \frac{K_1(p_1/R\rho_1)^{1/6} e^{K_2(p_1/R\rho_1)^{-1/3}}}{p_1(1 - e^{-R\Theta\rho_1/p_1})} \tag{12}$$

If the shock wave is described by $y_1 = f_1(x_1)$, the frozen situation behind the shock wave is given by the following shock conditions:

$$u_1[x_1,f_1(x_1)] = u_\infty \cos\beta \left\{1 + \frac{2[M_\infty^2 \sin^2(\theta+\beta) - 1]}{(\gamma+1)M_\infty^2} \times [\cot(\theta+\beta)\tan\beta - 1]\right\} \quad (13)$$

$$v_1[x_1,f_1(x_1)] = u_\infty \sin\beta \left\{-1 + \frac{2[M_\infty^2 \sin^2(\theta+\beta) - 1]}{(\gamma+1)M_\infty^2} \times [\cot(\theta+\beta)\cot\beta + 1]\right\} \quad (14)$$

$$p_1[x_1,f_1(x_1)] = p_\infty \left\{\frac{2\gamma M_\infty^2 \sin^2(\theta+\beta) - (\gamma-1)}{(\gamma+1)}\right\} \quad (15)$$

$$\rho_1[x_1,f_1(x_1)] = \rho_\infty \left\{\frac{(\gamma+1)M_\infty^2 \sin^2(\theta+\beta)}{(\gamma-1)M_\infty^2 \sin^2(\theta+\beta) + 2}\right\} \quad (16)$$

$$e_{v_1}[x_1,f_1(x_1)] = e_\infty \quad (17)$$

where β is the semivertex angle of the wedge, and θ is the shock angle (Fig 1), related to the shock shape through

$$\theta = \tan^{-1}f_1'(x_1) \quad (18)$$

in which the prime denotes differentiation with respect to the argument Freestream conditions are referred to by the subscript ∞ The tangency condition at the wedge surface is expressed by

$$v_1[x_1,0] = 0 \quad (19)$$

Finally, there is a condition expressing the fact that the shock wave is attached to the leading edge, namely,

$$f_1(0) = 0 \quad (20)$$

The system of equations and conditions (1–3, 7, and 10–20) defines completely the problem for the unknowns u_1, v_1, p_1, ρ_1, e_{v_1}, and $f_1(x_1)$

Perturbation Problem

The exact problem, owing to its complexity, will be solved approximately by making a perturbation about a known reference condition There are several such conditions to choose from For a slender wedge, the freestream would be a natural choice In the present case the wedge is not necessarily slender, and hence conditions existing behind the shock wave are more suitable for reference, in view of the assumptions that will be made subsequently The two most interesting conditions are the frozen state existing at the leading edge and the equilibrium state existing far downstream Because the equilibrium state cannot be given explicitly, the frozen state will be used here and will be denoted by subscript 0

Now introduce dimensionless variables appropriate to small perturbations of the basic frozen state by setting

$$u_1(x_1,y_1) = u_0[1 + u(x,y)] \quad (21)$$

$$v_1(x_1,y_1) = u_0 v(x,y) \quad (22)$$

$$p_1(x_1,y_1) = p_0[1 + p(x,y)] \quad (23)$$

$$\rho_1(x_1,y_1) = \rho_0[1 + \rho(x,y)] \quad (24)$$

$$e_1(x_1,y_1) = e_\infty + [\gamma/(\gamma-1)]RT_0 e(x,y) \quad (25)$$

where

$$x = x_1/u_0\tau_0 \quad (26)$$

$$y = y_1/u_0\tau_0 \quad (27)$$

and

$$f_1(x_1) = (\tan\theta_0)x_1 + u_0\tau_0 f(x) \quad (28)$$

so that

$$f_1'(x_1) = \tan\theta_0 + f'(x) \equiv \phi_0 + f'(x) \quad (29)$$

Here the reference values u_0, p_0, ρ_0, T_0, and θ_0 are given by the oblique shock-wave relations with the vibrational energy frozen,† and τ_0 is given by expression (12) with p_0 and ρ_0 in place of p_1 and ρ_1, respectively Note that e is referred to the enthalpy $\gamma RT_0/(\gamma-1)$ rather than to the vibrational energy $e_{v_0}(= e_\infty)$, which may be exponentially small; also note that there is no basic velocity in the y direction

Substituting Eqs (21–29) into the exact problem will lead to a problem for the perturbation quantities as functions of x and y Assuming that the perturbation quantities are small compared with unity in the region of interest, so that only linear terms are retained, yields the following first-order perturbation problem:

$$\frac{\partial\rho}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (30)$$

$$\frac{\partial u}{\partial x} + \frac{1}{\gamma M_0^2}\frac{\partial p}{\partial x} = 0 \quad (31)$$

$$\frac{\partial v}{\partial x} + \frac{1}{\gamma M_0^2}\frac{\partial p}{\partial y} = 0 \quad (32)$$

$$\frac{\partial p}{\partial x} - \gamma\frac{\partial\rho}{\partial x} + \gamma\frac{\partial e_v}{\partial x} = 0 \quad (33)$$

$$\frac{\partial e}{\partial x} - \frac{(\gamma-1)}{\gamma}\frac{e^{\Theta/T_0}(\Theta/T_0)^2}{(e^{\Theta/T_0}-1)^2}(p-\rho) + e = \frac{(\gamma-1)}{\gamma}\frac{(e_0^* - e_\infty)}{RT_0} \quad (34)$$

where

$$M_0^2 = u_0^2\rho_0/\gamma p_0$$

$$e_0^* = R\Theta/(e^{\Theta/T_0} - 1)$$

and

$$u(x,\phi_0 x) = 2\left\{\frac{M_\infty^2\cos^2\theta_0 \sin[2(\theta_0+\beta)][\cot(\theta_0+\beta)\tan\beta - 1]}{(\gamma+1)M_\infty^2 + 2[M_\infty^2\sin^2(\theta_0+\beta) - 1)][\cot(\theta_0+\beta)\tan\beta - 1]} - \frac{\{\sec^2\beta\tan\beta[M_\infty^2\sin^2(\theta_0+\beta) - 1]\}/(\tan\beta+\tan\theta_0)^2}{(\gamma+1)M_\infty^2 + 2[M_\infty^2\sin^2(\theta_0+\beta) - 1][\cot(\theta_0+\beta)\tan\beta - 1]}\right\} f'(x) \equiv K_u f'(x) \quad (35)$$

$$v(x,\phi_0 x) = 2\tan\beta\left\{\frac{M_\infty^2\cos^2\theta_0 \sin[2(\theta_0+\beta)][\cot(\theta_0+\beta)\cot\beta + 1]}{(\gamma+1)M_\infty^2 + 2[M_\infty^2\sin^2(\theta_0+\beta) - 1][\cot(\theta_0+\beta)\tan\beta - 1]} - \frac{\{\sec^2\beta\cot\beta[M_\infty^2\sin^2(\theta_0+\beta) - 1)]\}/(\tan\beta+\tan\theta_0)^2}{(\gamma+1)M_\infty^2 + 2[M_\infty^2\sin^2(\theta_0+\beta) - 1][\cot(\theta_0+\beta)\tan\beta - 1]}\right\} f'(x) \equiv K_v f'(x) \quad (36)$$

† u_0, p_0, and ρ_0 are given by the right-hand sides of Eqs (13, 15, and 16), respectively, with θ_0 in place of θ θ_0 is determined by solving for θ in Eq (14) with v_1 set equal to zero T_0 is given by $p_0/R\rho_0$

$$p(x,\phi_0 x) = \left\{\frac{2\gamma M_\infty^2 \cos^2\theta_0 \sin[2(\theta_0+\beta)]}{2\gamma M_\infty^2 \sin^2(\theta_0+\beta) - (\gamma-1)}\right\} f'(x) \equiv K_p f'(x) \tag{37}$$

$$\rho(x,\phi_0 x) = \left\{\frac{2\cos^2\theta_0 \sin[2(\theta_0+\beta)]}{\sin^2(\theta_0+\beta)[(\gamma-1)M_\infty^2 \sin^2(\theta_0+\beta)+2]}\right\} f'(x) \equiv K\rho f'(x) \tag{38}$$

$$e\,(x,\phi_0 x) = 0 \tag{39}$$

$$v(x,0) = 0 \tag{40}$$

$$f(0) = 0 \tag{41}$$

Physically, the assumption of small perturbations implies that nonequilibrium effects alter the flow field only slightly from the frozen solution (For example, the variation in pressure is only a small fraction of the pressure itself)

The perturbation velocity field will be shown to be rotational in the next section; therefore, a velocity potential does not exist for this problem However, if a stream function ψ is defined to satisfy Eq (30), a single partial differential equation of fourth order for ψ can be obtained in place of the system of equations (30–34) and is similar in form to Vincenti's equation for the velocity potential in his work on a slender wedge

Solution of the Perturbation Problem

Since Eqs (31) and (33) can be immediately integrated, it is convenient to deal with the system of equations (30–34), rather than with the single equation of fourth order for ψ Integrating Eq (31) and evaluating the constant of integration from the shock conditions gives

$$u + \frac{1}{\gamma M_0^2} p = \left(K_u + \frac{1}{\gamma M_0^2} K_p\right) f'\left(\frac{y}{\phi_0}\right) \tag{42}$$

which is the linearized Bernoulli's equation Proceeding similarly with Eq (33) yields

$$p - \gamma\rho + \gamma e_v = (K_p - \gamma K\rho) f'(y/\phi_0) \tag{43}$$

With the aid of Eqs (42) and (43), p and ρ can be eliminated from Eqs (30, 32, and 34); thus we obtain

$$\lambda_0^2 \frac{\partial u}{\partial x} - \frac{\partial e_v}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{44}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{1}{\phi_0}\left(K_u + \frac{1}{\gamma M_0^2} K_p\right) f''\left(\frac{y}{\phi_0}\right) \tag{45}$$

$$\frac{\partial e_v}{\partial x} + \left\{1 + \frac{(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2}\right\} e_v + \frac{(\gamma-1)^2 e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2} M_0^2 u = \frac{(\gamma-1)}{\gamma}\frac{(e_0^* - e_\infty)}{RT_0} + \frac{(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2}[(\gamma-1)M_0^2 K_u + K_p - K\rho] f'\left(\frac{y}{\phi_0}\right) \tag{46}$$

where $\lambda_0^2 \equiv (M_0^2 - 1)$ The vorticity of the perturbation velocity field is now given by Eq (45) and is seen to be proportional, for a given streamline, to the curvature of the shock wave where the streamline crosses the shock wave

In order to use the Laplace transformation to solve the last three equations, a change in the coordinate system is made for convenience Set

$$\left.\begin{aligned}\xi &= x - (y/\phi_0)\\ \eta &= y/\phi_0\end{aligned}\right\} \tag{47}$$

Then Eqs (44–46) become

$$\lambda_0^2 \frac{\partial u}{\partial \xi} + \frac{1}{\phi_0}\left(\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta}\right) - \frac{\partial e_v}{\partial \xi} = 0 \tag{48}$$

$$\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} - \phi_0 \frac{\partial v}{\partial \xi} = \left(K_u + \frac{1}{\gamma M_0^2} K_p\right) f''(\eta) \tag{49}$$

$$\frac{\partial e_v}{\partial \xi} + \left\{1 + \frac{(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2}\right\} e + \frac{(\gamma-1)^2 e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2} M_0^2 u = \frac{(\gamma-1)}{\gamma}\frac{(e_0^* - e_\infty)}{RT_0} + \frac{(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2}[(\gamma-1)M_0^2 K_u + K_p - K\rho] f'(\eta) \tag{50}$$

Also, conditions (35, 36, 39, and 40) can be written as

$$u(0,\eta) = K_u f'(\eta) \tag{51}$$

$$v(0,\eta) = K_v f'(\eta) \tag{52}$$

$$e\,(0,\eta) = 0 \tag{53}$$

$$v(\xi,0) = 0 \tag{54}$$

Let $\bar{u}$, $\bar{v}$, and $\bar{e}_v$ denote, respectively, the Laplace transforms of u, v, and e_v with respect to ξ; for example,

$$L[u(\xi,\eta)] = \int_0^\infty e^{-s\xi} u(\xi,\eta) d\xi = \bar{u}(\eta; s)$$

where s is the parameter of the transformation The problem defined by expressions (48–54) can then be transformed into

$$\lambda_0^2[s\bar{u} - K_u f'(\eta)] - \frac{1}{\phi_0}\left[\frac{d\bar{v}}{d\eta} - s\bar{v} + K_v f'(\eta)\right] - s\bar{e}_v = 0 \tag{55}$$

$$\left[\frac{d\bar{u}}{d\eta} - s\bar{u} + K_u f'(\eta)\right] - \phi_0 [s\bar{v} - K_v f'(\eta)] = \frac{1}{s}\left(K_u + \frac{1}{\gamma M_0^2} K_p\right) f''(\eta) \tag{56}$$

$$s\bar{e}_v + \left\{1 + \frac{(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2}\right\} \bar{e}_v + \frac{(\gamma-1)^2 e^{\Theta/T_0}(\Theta/T_0)^2}{\gamma(e^{\Theta/T_0}-1)^2} M_0^2 \bar{u} = \frac{(\gamma-1)}{\gamma}\frac{(e_{v_0}^* - e_\infty)}{sRT_0} + \frac{(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2}{s\gamma(e^{\Theta/T_0}-1)^2} \times [(\gamma-1)M_0^2 K_u + K_p - K\rho] f'(\eta) \tag{57}$$

$$\bar{v}(0) = 0 \tag{58}$$

The three simultaneous equations (55–57) can be solved to give the solutions for $\bar{v}$, $\bar{u}$, and $\bar{e}_v$ With the aid of Eqs (42) and (43), $\bar{p}$ and $\bar{\rho}$ can also be obtained

If $\sigma(s)$ is defined by

$$\sigma(s) \equiv \frac{(\gamma-1)^2 M_0^2 \phi_0^2}{\gamma\{1 + s + [(\gamma-1)e^{\Theta/T_0}(\Theta/T_0)^2/\gamma(e^{\Theta/T_0}-1)^2]\}} \times \frac{e^{\Theta/T_0}(\Theta/T_0)^2}{(e^{\Theta/T_0}-1)^2} \tag{59}$$

then

$$\bar{v} = \frac{1}{2[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} \left\{\left[\frac{\lambda_0^2\phi_0}{\gamma M_0^2} K_p + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}K_v + \frac{\sigma(s)}{(\gamma - 1)M_0^2\phi_0}\left(\frac{K_p - K_\rho}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} - \frac{K_p}{\gamma} + K_\rho\right)\right] \times \right.$$
$$\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi - \left[\frac{\lambda_0^2\phi_0}{\gamma M_0^2} K_p - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}K + \right.$$
$$\left.\left.\frac{\sigma(s)}{(\gamma - 1)M_0^2\phi_0}\left(\frac{K_p - K_\rho}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} - \frac{K_p}{\gamma} + K_\rho\right)\right] \int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi\right\} +$$
$$\frac{(\gamma - 1)}{\gamma}\frac{(e_{v_0}{}^* - e_{v_\infty})}{RT_0} \frac{\phi_0}{[1 - \lambda_0^2\phi_0^2 - \sigma(s)]s(1 + s)\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} \tag{60}$$

$$\bar{u} = -\frac{1}{2[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} \left\{\left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p + \phi_0 K + \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times \right.$$
$$\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi + \left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p - \phi_0 K - \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times$$
$$\left.\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi\right\} - \frac{(\gamma - 1)}{\gamma}\frac{(e_{\ 0}{}^* - e_{v_\infty})}{RT_0} \times$$
$$\frac{\phi_0^2}{[1 - \lambda_0^2\phi_0^2 - \sigma(s)]s(1 + s)\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} + \frac{1}{s}\left[K_u + \left(\frac{1}{\gamma M_0^2}\right)K_p\right] f'(\eta) \tag{61}$$

$$\bar{p} = \frac{\gamma M_0^2}{2[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} \left\{\left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p + \phi_0 K_v + \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times \right.$$
$$\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi + \left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p - \phi_0 K - \right.$$
$$\left.\left.\frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right]\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi\right\} +$$
$$\frac{(\gamma - 1)}{\gamma}\frac{(e_{\ 0}{}^* - e_{\ \infty})}{RT_0} \frac{\gamma M_0^2\phi_0^2}{[1 - \lambda_0^2\phi_0^2 - \sigma(s)]s(1 + s)\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} \tag{62}$$

$$\bar{e}_v = \frac{\sigma(s)}{2\phi_0^2[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} \left\{\left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p + \phi_0 K + \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times \right.$$
$$\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi + \left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p - \phi_0 K - \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times$$
$$\left.\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi\right\} + \frac{(\gamma - 1)}{\gamma}\frac{(e_{\ 0}{}^* - e_{\ \infty})}{RT_0} \times$$
$$\frac{(1 - \lambda_0^2\phi_0^2)}{[1 - \lambda_0^2\phi_0^2 - \sigma(s)]s(1 + s)\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} + \frac{\sigma(s)(K_p/\gamma - K_\rho)}{(\gamma - 1)M_0^2\phi_0^2 s} f'(\eta) \tag{63}$$

$$\bar{\rho} = \frac{[M_0^2\phi_0^2 + \sigma(s)]}{2\phi_0^2[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} \left\{\left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p + \phi_0 K + \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times \right.$$
$$\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi + \left[\frac{[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}{\gamma M_0^2} K_p - \phi_0 K - \frac{\sigma(s)}{(\gamma - 1)M_0^2}\frac{(K_p - K_\rho)}{\{1 + [\lambda_0\phi_0^2 + \sigma(s)]^{1/2}\}}\right] \times$$
$$\left.\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} + \eta\right) d\xi\right\} + \frac{(\gamma - 1)}{\gamma}\frac{(e_{\ 0}{}^* - e_{\ \infty})}{RT_0} \times$$
$$\frac{(1 + \phi_0^2)}{[1 - \lambda_0^2\phi_0^2 - \sigma(s)]s(1 + s)\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} - \frac{(K_p/\gamma - K_\rho)}{s\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} f'(\eta) \tag{64}$$

Finally, the unknown slope of the shock wave, $f'(\eta)$, is found by enforcing condition (58); thus we obtain

$$\frac{1}{2[\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} \left\{\left[\frac{\lambda_0^2\phi_0}{\gamma M_0^2} K_p + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}K_v + \frac{\sigma(s)}{(\gamma - 1)M_0^2\phi_0}\left(\frac{K_p - K_\rho}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} - \frac{K_p}{\gamma} + K_\rho\right)\right] \times \right.$$
$$\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}\right) d\xi - \left[\frac{\lambda_0^2\phi_0}{\gamma M_0^2} K_p - [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}K + \frac{\sigma(s)}{(\gamma - 1)M_0^2\phi_0} \times \right.$$
$$\left.\left.\left(\frac{K_p - K_\rho}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}} - \frac{K_p}{\gamma} + K_\rho\right)\right] \int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1 + [\lambda_0^2\phi_0^2 + \sigma(s)]^{1/2}}\right) d\xi\right\} +$$
$$\frac{(\gamma - 1)}{\gamma}\frac{(e_{\ 0}{}^* - e_{\ \infty})}{RT_0} \frac{\phi_0}{[1 - \lambda_0^2\phi_0^2 - \sigma(s)]s(1 + s)\{1 + \sigma(s)/[(\gamma - 1)M_0^2\phi_0^2 - \sigma(s)]\}} = 0 \tag{65}$$

The solution of this equation, together with condition (41), determines the shock shape

Expressions (60–65) complete the solution of the perturbation problem in the transformed form However, the complexity of these expressions with respect to s forbids an analytical inversion of the transform; therefore, further simplifications are needed before the picture of the flow field can be seen explicitly

Survey of Realizations of Small-Perturbation Assumption

The small-perturbation assumption, made in obtaining the first-order perturbation problem, is a crucial step in the analysis; therefore, the circumstances under which it can be realized deserve a survey The perturbation problem as defined by equations and conditions (30–41) has only one nonhomogeneous term, namely, the right-hand side of Eq (34) (Had the equilibrium situation existing far downstream of the leading edge been chosen as the reference state, then the nonhomogeneity of the resulting perturbation problem would exist in the shock conditions instead) As the magnitude of all perturbation quantities is proportional to that of the nonhomogeneous term, it appears to be sufficient to have this term small in order to realize the small-perturbation assumptions

This term can be expressed in terms of freestream quantities as

$$\frac{(\gamma-1)}{\gamma}\frac{(e_{v_0}{}^* - e_{v\infty})}{RT_0} = \frac{(\gamma-1)}{\gamma}\left(\frac{T_\infty}{T_0}\right)\times \left[\frac{\Theta/T_\infty}{e^{(\Theta/T_\infty)(T_\infty/T_0)}-1} - \frac{e_{v\infty}}{RT_\infty}\right]$$

where

$$\frac{T_\infty}{T_0} = \frac{(\gamma+1)^2 M_\infty{}^2\sin^2(\theta_0+\beta)}{[2\gamma M_\infty{}^2\sin^2(\theta_0+\beta)-(\gamma-1)]\,[(\gamma-1)M_\infty{}^2\sin^2(\theta_0+\beta)+2]}$$

This expression involves the following four dimensionless parameters (three if the freestream is in equilibrium):

$$\gamma \qquad \Theta/T_\infty \qquad M_\infty{}^2\sin^2(\theta_0+\beta)$$

$$\frac{e_{v\infty}}{RT_\infty}\left(= \frac{(\Theta/T_\infty)}{e^{(\Theta/T_\infty)}-1}\ \text{if the freestream is in equilibrium}\right)$$

Considering possible limits for these various parameters shows that this term vanishes in the following cases:

1) $\gamma \to 1$: Since the enthalpy with the vibrational energy unexcited is $[\gamma/(\gamma-1)]RT(=\frac{7}{2}RT$ for a diatomic gas), whereas the fully excited vibrational energy is only RT, this limiting case implies that the full vibrational energy is small compared with the enthalpy

2) $(\Theta/T_\infty) \to \infty$ and $(e_{v\infty}/RT_\infty) \to 0$ (the latter limit is not necessary if the freestream is in equilibrium): This limit implies that the temperature is low compared with the characteristic vibrational temperature and, hence, that the excited vibrational energy is small compared with the enthalpy

These two cases correspond to Spence's assumption,[6] although he may have had particularly the second case in mind

3) $M_\infty{}^2\sin^2(\theta_0+\beta) \to 1$ and $(e_{\ \infty}/RT_\infty) \to [(\Theta/T_\infty)/(e^{\Theta/T_\infty}-1)]$: This is the case of a slender wedge, which can create only a small disturbance

4) $(e_{v\infty}/RT_\infty) \to (\Theta/T_\infty)/[e^{(\Theta/T_\infty)(T_\infty/T_0)}-1]$: This is the case when the vibrational energy in the freestream is excited to almost the equilibrium value behind the shock wave at the leading edge

The preceding conditions are sufficient for the perturbations to be small, but they are not necessary If uniform smallness of the perturbation is not required, then there is another useful case involving a coordinate rather than a parameter perturbation:

5) $x \to 0$ and $y \to 0$: This means that the region of interest is limited to distances from the leading edge small compared with the reference relaxation length (Note that the frozen-flow quantities used as references do actually exist at the leading edge)

Simplification of the Results

Specializing the general small-perturbation assumption to cases 1, 2, 3, and 5 as listed in the last section simplifies the perturbation solution (60–65) Of particular interest are cases 1 and 2, where expressions (60–65) reduce to such simple form that they can be inverted analytically to give an explicit solution for the entire flow field behind the shock wave Specializing Eqs (30–34) of the general perturbation problem shows that in these two cases the perturbation rate equation is uncoupled from the other perturbation equations and could readily have been integrated directly

Equation (59) shows that $\sigma(s)$ is small in both cases 1 and 2, and thus the terms associated with it in expressions (60–65) are negligible With the inverse transformations

$$L^{-1}\left\{\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1-\lambda_0\phi_0}+\eta\right)d\xi\right\} = f'\left(\frac{\xi}{1-\lambda_0\phi_0}+\eta\right)$$

$$L^{-1}\left\{\int_0^\infty e^{-s\xi} f'\left(\frac{\xi}{1+\lambda_0\phi_0}+\eta\right)d\xi\right\} = f'\left(\frac{\xi}{1+\lambda_0\phi_0}+\eta\right)$$

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s(1+s)}\right\} = (1-e^{-\xi})$$

$$L^{-1}\{\bar{v}\} = v$$

etc , and Eqs (47), expressions (60–65) yield

$$v = \frac{1}{2}\left[\left(K_v + \frac{\lambda_0}{\gamma M_0{}^2}K_p\right)f'\left(\frac{x-\lambda_0 y}{1-\lambda_0\phi_0}\right) + \left(K_v - \frac{\lambda_0}{\gamma M_0{}^2}K_p\right)f'\left(\frac{x+\lambda_0 y}{1+\lambda_0\phi_0}\right)\right] + \frac{(\gamma-1)}{\gamma}\times \frac{(e_{v_0}{}^* - e_{\ \infty})}{RT_0}\frac{\phi_0}{(1-\lambda_0{}^2\phi_0{}^2)}\left\{1-\exp\left[-\left(x-\frac{y}{\phi_0}\right)\right]\right\} \qquad (66)$$

$$u = -\frac{1}{2\lambda_0}\left[\left(K_v + \frac{\lambda_0}{\gamma M_0{}^2}K_p\right)f'\left(\frac{x-\lambda_0 y}{1-\lambda_0\phi_0}\right) - \left(K_v - \frac{\lambda_0}{\gamma M_0{}^2}K_p\right)f'\left(\frac{x+\lambda_0 y}{1+\lambda_0\phi_0}\right)\right] - \frac{(\gamma-1)}{\gamma}\frac{(e_{\ 0}{}^* - e_{\ \infty})}{RT_0}\frac{\phi_0{}^2}{(1-\lambda_0{}^2\phi_0{}^2)}\times \left\{1-\exp\left[-\left(x-\frac{y}{\phi_0}\right)\right]\right\} + \left(K_u + \frac{K_p}{\gamma M_0{}^2}\right)f'\left(\frac{y}{\phi_0}\right) \qquad (67)$$

$$p = \frac{\gamma M_0{}^2}{2\lambda_0}\left[\left(K_v + \frac{\lambda_0}{\gamma M_0{}^2}K_p\right)f'\left(\frac{x-\lambda_0 y}{1-\lambda_0\phi_0}\right) - \left(K_{\ } - \frac{\lambda_0}{\gamma M_0{}^2}K_p\right)f'\left(\frac{x+\lambda_0 y}{1+\lambda_0\phi_0}\right)\right] + \frac{(\gamma-1)}{\gamma}\times \frac{(e_{\ 0}{}^* - e_{\ \infty})}{RT_0}\frac{\gamma M_0{}^2\phi_0{}^2}{(1-\lambda_0{}^2\phi_0{}^2)}\left\{1-\exp\left[-\left(x-\frac{y}{\phi_0}\right)\right]\right\} \qquad (68)$$

$$e = \frac{(\gamma - 1)}{\gamma}\frac{(e_0{}^* - e_\infty)}{RT_0}\left\{1 - \exp\left[-\left(x - \frac{y}{\phi_0}\right)\right]\right\} \quad (69)$$

$$p = \frac{M_0^2}{2\lambda_0}\left[\left(K + \frac{\lambda_0}{\gamma M_0^2}K_p\right)f'\left(\frac{x - \lambda_0 y}{1 - \lambda_0\phi_0}\right) - \left(K - \frac{\lambda_0}{\gamma M_0^2}K_p\right)f'\left(\frac{x + \lambda_0 y}{1 + \lambda_0\phi_0}\right)\right] + \frac{(\gamma - 1)}{\gamma}\frac{(e_0{}^* - e_\infty)}{RT_0}\frac{(1 + \phi_0^2)}{(1 - \lambda_0^2\phi_0^2)} \times \left\{1 - \exp\left[-\left(x - \frac{y}{\phi_0}\right)\right]\right\} + \left(K_\rho - \frac{K_p}{\gamma}\right)f'\left(\frac{y}{\phi_0}\right) \quad (70)$$

$$\frac{1}{2}\left[\left(K + \frac{\lambda_0}{\gamma M_0^2}K_p\right)f'\left(\frac{x}{1 - \lambda_0\phi_0}\right) + \left(K - \frac{\lambda_0}{\gamma M_0^2}K_p\right)f'\left(\frac{x}{1 + \lambda_0\phi_0}\right)\right] + \frac{(\gamma - 1)}{\gamma}\frac{(e_{v_0}{}^* - e_\infty)}{RT_0}\frac{\phi_0}{(1 - \lambda_0^2\phi_0^2)}[1 - e^{-x}] = 0 \quad (71)$$

It can be seen from the last section that the physical significance of cases 1 and 2 is quite parallel; therefore, unsurprisingly, the perturbation solution for these two physically distinct cases takes the same mathematical form

This solution consists, in general, of four terms, and each of them can be interpreted physically Consider, for instance, Eq (67) for u The first term is constant on lines $x - \lambda_0 y =$ const, which are the frozen Mach lines leaving the wedge surface; therefore, the first term represents the propagation of disturbance along the outgoing frozen Mach lines Similarly, the second term represents the propagation of disturbances along the incoming frozen Mach lines The third term describes the relaxation of the gas, i e , the behavior of exponential decay, from the position of the frozen-flow shock wave The fourth term is the one missing from all existing approximate solutions,‡ although its existence is evident from both numerical solutions and physical reasoning It describes the fact that, far downstream of the leading edge, i e , at very large x, although the first three terms give constant values, the flow quantities have different values on lines $y =$ const, which are the basic streamlines This effect is referred to as an entropy layer

Each of the terms in Eq (67) can be traced back to its antecedent in Eq (61) before the inversion is carried out However, the entropy-layer term is absent from the corresponding Eqs (60, 62, and 65) and, hence, also from the inverted solution (66, 68, and 71) for the normal velocity, pressure, and shock shape Although all four effects are present in Eq (63), only the relaxation term survives the approximation that permits carrying out the inversion for the vibrational energy (69) The constancy of pressure across the entropy layer is analogous to that across a viscous boundary layer

The functional equation (71) can be rewritten as

$$f'(x) + \left(\frac{\gamma M_0^2 K - \lambda_0 K_p}{\gamma M_0^2 K_v + \lambda_0 K_p}\right)f'\left(\frac{1 - \lambda_0\phi_0}{1 + \lambda_0\phi_0}x\right) + 2\frac{(\gamma - 1)}{\gamma}\frac{(e_0{}^* - e_\infty)}{RT_0}\frac{\gamma M_0^2\phi_0}{(\gamma M_0^2 K_v + \lambda_0 K_p)(1 - \lambda_0^2\phi_0^2)} \times [1 - e^{-(1 - \lambda_0\phi_0)x}] = 0 \quad (72)$$

The physical interpretation of this relation as representing repeated reflections of disturbances from the shock wave has been well explained in detail by Chu[10] Chu treated the perturbation resulting from a slight deformation of the wedge surface in classical gasdynamics so that his disturbance is of mechanical origin, whereas in the present problem it is of thermodynamic origin It may be of interest to point out that the coefficient of the second term, explained by Chu as an amplifying factor, can be either positive or negative in this case To solve Eq (72), an operator P is defined such that

$$PF(x) = F\left(\frac{1 - \lambda_0\phi_0}{1 + \lambda_0\phi_0}x\right)$$

where $F(x)$ is any function of x Operating on Eq (72) yields

$$f'(x) = \frac{-2[(\gamma - 1)/\gamma][(e_0{}^* - e_\infty)/RT_0][\gamma M_0^2\phi_0/(\gamma M_0^2 K + \lambda_0 K_p)(1 - \lambda_0^2\phi_0^2)][1 - e^{(-1-\lambda_0\phi_0)x}]}{1 + [(\gamma M_0^2 K_v - \lambda_0 K_p)/(\gamma M_0^2 K_v + \lambda_0 K_p)]P}$$

Expanding the foregoing expression in an infinite series then gives

$$f'(x) = \frac{(\gamma - 1)}{\gamma}\frac{(e_0{}^* - e_\infty)}{RT_0}\frac{\phi_0}{K(1 - \lambda_0^2\phi_0^2)} \times \left\{-1 + \frac{2\gamma M_0^2 K_v}{(\gamma M_0^2 K + \lambda_0 K_p)}\sum_{i=0}^{\infty}(-1)^i\left(\frac{\gamma M_0^2 K_v - \lambda_0 K_p}{\gamma M_0^2 K + \lambda_0 K_p}\right)^i \times \exp\left[-\left(\frac{1 - \lambda_0\phi_0}{1 + \lambda_0\phi_0}\right)^i(1 - \lambda_0\phi_0)x\right]\right\} \quad (73)$$

from which it can be shown that the shock wave starts with the frozen slope and curves either forward or backward to its equilibrium slope at infinity, depending on whether e_∞ is larger or smaller than $e_0{}^*$ Integrating Eq (73) and using condition (41) gives

$$f(x) = \frac{(\gamma - 1)}{\gamma}\frac{(e_{v_0}{}^* - e_\infty)}{RT_0}\frac{\phi_0}{K_v(1 - \lambda_0^2\phi_0^2)} \times \left\{-x + \frac{2\gamma M_0^2 K_v}{(\gamma M_0^2 K_v + \lambda_0 K_p)(1 - \lambda_0\phi_0)} \times \left[\frac{(\gamma M_0^2 K_v + \lambda_0 K_p)(1 - \lambda_0\phi_0)}{(\gamma M_0^2 K + \lambda_0 K_p)(1 - \lambda_0\phi_0) + (\gamma M_0^2 K - \lambda_0 K_p)(1 + \lambda_0\phi_0)} - \sum_{i=0}^{\infty}(-1)^i\left(\frac{(\gamma M_0^2 K - \lambda_0 K_p)(1 + \lambda_0\phi_0)}{(\gamma M_0^2 K_v + \lambda_0 K_p)(1 - \lambda_0\phi_0)}\right)^i \times \exp\left(-\left(\frac{1 - \lambda_0\phi_0}{1 + \lambda_0\phi_0}\right)^i(1 - \lambda_0\phi_0)x\right)\right]\right\} \quad (74)$$

The constant second term represents the displacement of the shock wave ahead of its equilibrium position far from the wedge

For case 3, Eqs (60–65) should lead to the result of Ref 3 (and Ref 2) for the slender wedge, but the formal reduction is too cumbersome to be carried out It can be shown, how-

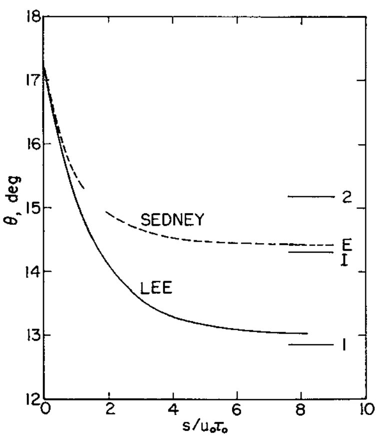

Fig 2 Shock wave angle vs nondimensional arc length along shock wave

‡ An exception is the recent work of Zhigulev,[11] who arrives at the same result by applying Spence s assumption to the exact problem directly at the outset

ever, that the perturbation problem itself, as defined by Eqs (30–41), is completely equivalent to Eqs (5–12 and 31–37) of Ref 3 For case 4, no simplification can be made in Eqs (60–65) The physical significance of these two cases may be considered as somewhat parallel, and so mathematically in both cases the perturbation rate equation is coupled with the other perturbation equations, and explicit solution cannot be expected

Case 5 corresponds to the limit $s \to \infty$, and thus terms with $\sigma(s)$ can again be neglected in Eqs (60–65) Also, unity is negligible compared with s With the inverse transformations given before, together with $L^{-1}\{1/s^2\} = \xi$ and Eqs (47), expressions (60–65) now yield

$$v = \frac{1}{2}\left[\left(K_v + \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x - \lambda_0 y}{1 - \lambda_0 \phi_0}\right) + \left(K_v - \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x + \lambda_0 y}{1 + \lambda_0 \phi_0}\right)\right] + \frac{(\gamma - 1)}{\gamma} \frac{(e_{v_0}^* - e_{\infty})}{RT_0} \frac{\phi_0}{(1 - \lambda_0^2 \phi_0^2)}\left(x - \frac{y}{\phi_0}\right) \quad (75)$$

$$u = -\frac{1}{2\lambda_0}\left[\left(K_v + \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x - \lambda_0 y}{1 - \lambda_0 \phi_0}\right) - \left(K_v - \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x + \lambda_0 y}{1 + \lambda_0 \phi_0}\right)\right] + \left(K_u + \frac{K_p}{\gamma M_0^2}\right) f'\left(\frac{y}{\phi_0}\right) - \frac{(\gamma - 1)}{\gamma} \frac{(e_{0}^* - e_{\infty})}{RT_0} \times \frac{\phi_0^2}{(1 - \lambda_0^2 \phi_0^2)}\left(x - \frac{y}{\phi_0}\right) \quad (76)$$

$$p = \frac{\gamma M_0^2}{2\lambda_0}\left[\left(K + \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x - \lambda_0 y}{1 - \lambda_0 \phi_0}\right) - \left(K_v - \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x + \lambda_0 y}{1 + \lambda_0 \phi_0}\right)\right] + \frac{(\gamma - 1)}{\gamma} \frac{(e_{v_0}^* - e_{\infty})}{RT_0} \frac{\gamma M_0^2 \phi_0^2}{(1 - \lambda_0^2 \phi_0^2)}\left(x - \frac{y}{\phi_0}\right) \quad (77)$$

$$e_v = \frac{(\gamma - 1)}{\gamma} \frac{(e_{v_0}^* - e_{v\infty})}{RT_0}\left(x - \frac{y}{\phi_0}\right) \quad (78)$$

$$\rho = \frac{M_0^2}{2\lambda_0}\left[\left(K_v + \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x - \lambda_0 y}{1 - \lambda_0 \phi_0}\right) - \left(K_v - \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x + \lambda_0 y}{1 + \lambda_0 \phi_0}\right)\right] + \left(K_\rho - \frac{K_p}{\gamma}\right) f'\left(\frac{y}{\phi_0}\right) + \frac{(\gamma - 1)}{\gamma} \frac{(e_{v_0}^* - e_{\infty})}{RT_0} \times \frac{(1 + \phi_0^2)}{(1 - \lambda_0^2 \phi_0^2)}\left(x - \frac{y}{\phi_0}\right) \quad (79)$$

$$\frac{1}{2}\left[\left(K + \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x}{1 - \lambda_0 \phi_0}\right) + \left(K_v - \frac{\lambda_0}{\gamma M_0^2} K_p\right) f'\left(\frac{x}{1 + \lambda_0 \phi_0}\right)\right] + \frac{(\gamma - 1)}{\gamma} \frac{(e_{0}^* - e_{v\infty})}{RT_0} \frac{\phi_0}{(1 - \lambda_0^2 \phi_0^2)} x = 0 \quad (80)$$

The solution of Eq (80) is

$$f'(x) = -\frac{(\gamma - 1)}{\gamma} \frac{(e_{0}^* - e_{v\infty})}{RT_0} \frac{\gamma M_0^2 \phi_0}{(\gamma M_0^2 K + \lambda_0^2 \phi_0 K_p)} x \quad (81)$$

which yields, after integrating and imposing condition (41),

$$f(x) = -\frac{(\gamma - 1)}{2\gamma} \frac{(e_{v_0}^* - e_{\infty})}{RT_0} \frac{\gamma M_0^2 \phi_0}{(\gamma M_0^2 K_v + \lambda_0^2 \phi_0 K_p)} x^2 \quad (82)$$

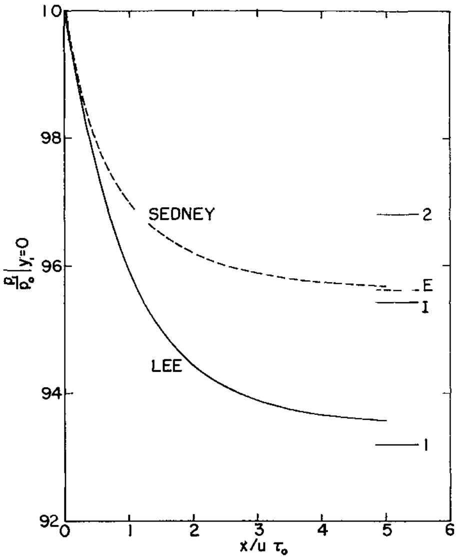

Fig 3 Nondimensional pressure vs nondimensional distance along wedge surface

As pointed out in the last section, results (75–82) are good approximations only when x and y are small compared with unity

Because of the hyperbolic nature of the governing differential equations and the fact that the perturbation is about the foremost point of the flow field, both the parameter and coordinate perturbations give the exact initial gradient of the flow quantities at the leading edge and the exact initial curvature of the shock wave Thus the parameter perturbation of cases 1 and 2 contains the coordinate perturbation of case 5 as a special case This can be shown by expanding the exponential functions in expressions (66–74) and keeping only the lowest-order terms in x and y It can be expected, therefore, that the parameter perturbation of cases 1 and 2 will be most accurate near the leading edge

Numerical Example

The following example has been treated using the solution of the parameter perturbation as given by Eqs (66–74): gas, N_2 with $\Theta = 3350°K$; wedge, $\beta = 40°$; freestream, in equilibrium; $M_\infty = 6$; $T_\infty = 300°K$; and $p_\infty = 1$ atm The results are presented in nondimensional form in Figs

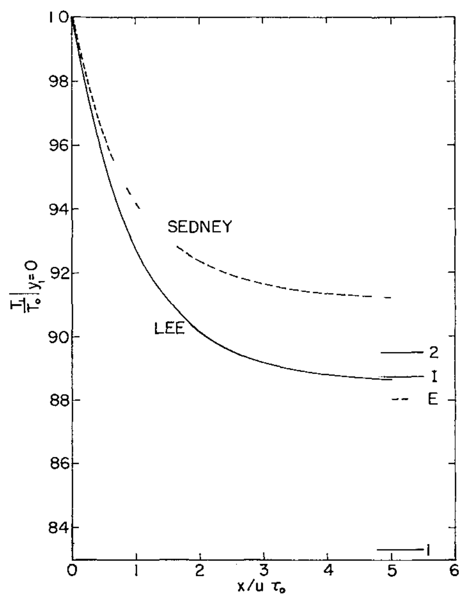

Fig. 4 Nondimensional temperature vs nondimensional distance along wedge surface

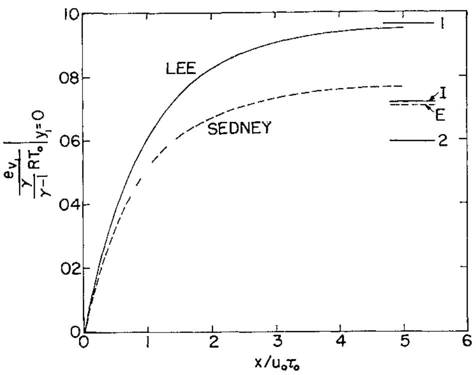

Fig 5 **Nondimensional vibrational energy vs nondimensional distance along wedge surface**

2–8 in comparison with the numerical solution of the exact problem computed by Sedney, South, and Gerber[8] using the method of characteristics Figure 2 plots the variation of shock angle with arc length along the shock wave Figures 3–6 show the variations of pressure, temperature, vibrational energy, and speed, respectively, with distance along the wedge surface Figures 7 and 8 present the variations of pressure and temperature, respectively, with distance normal to the wedge surface far downstream of the leading edge

The equilibrium-flow values of flow quantities existing downstream in the equilibrium core (Fig 1) can be calculated from the equilibrium shock relations These values are shown by lines marked E in Figs 2–5 The equilibrium shock relations can also be solved approximately, consistent with the small-perturbation assumption leading to the results given by Eqs (60–65) These approximate values are represented by lines marked I in Figs 2–5 Also, approximate equilibrium-flow values corresponding to the perturbation solution given by Eqs (66–74) are shown by lines marked 1 In obtaining these values, terms of higher order in the small parameter of cases 1 and 2 are neglected If these terms are accounted for by iterating, the resulting shock relations yield a second approximation, which is shown by lines marked 2

Because of the presence of the entropy layer, the curves in Figs 4–6 do not approach their corresponding equilibrium-flow values Figures 7 and 8 show that the present analysis gives a good indication of the thicknesses of the relaxation zone and entropy layer far downstream of the leading edge

In general, the results calculated by the perturbation solution agree qualitatively with the numerical solution Since the lines marked I are much closer to the lines marked E than the lines marked 1, it can be expected that the results would be in better quantitative agreement had they

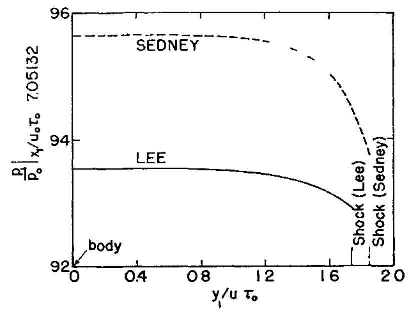

Fig 7 **Nondimensional pressure vs nondimensional distance normal to wedge surface at $x_1/u_0\tau_0$ = 7 05132**

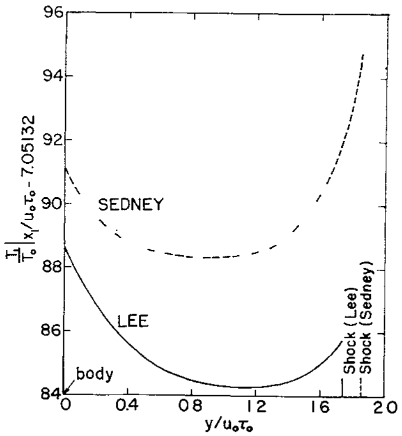

Fig 8 **Nondimensional temperature vs nondimensional distance normal to wedge surface at $x_1/u_0\tau_0$ = 7 05132**

been calculated by numerical integration from Eqs (60–65) The possible improvement from going to the second approximation of the parameter perturbation cases 1 and 2 is indicated by the fact that the lines marked 2 are about one-half as far from the lines marked E as the lines marked 1

Among the flow quantities, the temperature and the vibrational energy are most sensitive to nonequilibrium effects, whereas the others, especially the speed, are less sensitive and thus have a larger percentage of error However, in every figure the discrepancy between the perturbation solution and the numerical solution is at most of the order of square of the forementioned nonhomogeneous term, which, in this case, is about 0 1, in consistency with the analysis

Also, as pointed out in the last section, the results agree best with the numerical solution at points close to the leading edge It is reasonable to expect that, if the equilibrium situation existing far downstream in the equilibrium core were used as the reference in the parameter perturbation, the results would agree best at points far from the leading edge In practical applications, it might be advantageous to use both approaches to complement one another

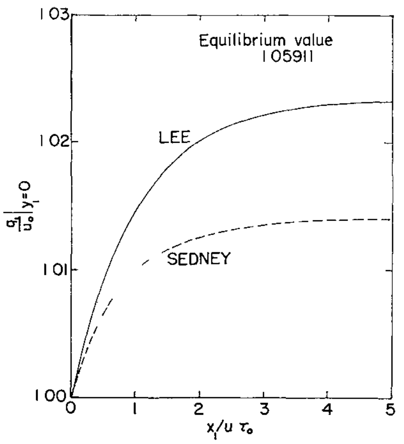

Fig 6 **Nondimensional speed vs nondimensional distance along wedge surface**

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Lift on an Oscillating Ellipsoid of Revolution

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A theoretical and experimental investigation of the lift and moment on an $\frac{8}{1}$ ellipsoid of revolution performing small heaving oscillations has been carried out The maximum circulatory lift coefficient measured was 0 035, about half the value obtained in steady flow at the same maximum angle of attack and Reynolds number Also, a phase lag of 20° in the buildup of the lift was observed The maximum pitching moment was reduced by 16% from the value predicted by potential theory The circulatory lift was calculated theoretically by estimating the net vorticity discharged into the main body of the fluid from the boundary layer at any instant of time This component of lift is caused by a vector circulation on a control surface surrounding the body and its wake The total lift was obtained by adding to the circulatory lift two additional components that depend on acceleration and are associated with the "apparent mass" of the body

Nomenclature

A	= amplitude of oscillation of the model
a	= half-length of major axis of the model
c_p	= pressure coefficient = $(p - p)/(\frac{1}{2}\rho V_\infty^2)$
h_i	= metric coefficients (i = 1, 2, 3)
i,j,k	= unit vectors
n	= unit normal
p	= pressure
RN	= Reynolds number = $V_\infty a/\nu$
t	= time
U,W,V	= velocity components of the external flow in directions x_i (i = 1, 2, 3, respectively)
$\mathbf{V}_\infty$	= freestream velocity
(s,z,y)	= coordinate system along streamlines and equipotential lines of the mean external flow, y being normal to the surface of the body
(x_1,x_2,x_3)	= orthogonal curvilinear coordinates; x_1 and x_2 are body surface coordinates, x_3 is normal to the surface
δ	= boundary-layer thickness
ν	= kinematic viscosity of fluid
ρ	= density of fluid
ω	= frequency of oscillation of the model
Ω	= angle of meridian from vertical plane

Subscripts

1	= fluctuating component
u,w,v	= velocity components in boundary layer in directions x_i (i = 1, 2, 3, respectively)
δ^*	= displacement thickness of the boundary layer
θ	= momentum thickness of the boundary layer

Introduction

IN Ref 1 a general theory concerning the forces acting on a body in incompressible, viscous, unsteady flow was presented It was shown that the instantaneous lift force could be predicted in terms of a vector circulation on an arbitrarily large control surface surrounding the body and its wake and that, in addition, two terms proportional to acceleration had to be included

The present paper describes an experimental investigation on a slender ellipsoid of revolution performing small heaving oscillations in a steady stream The objectives of these tests consisted in measuring instantaneous lift forces and pitching moments caused by viscous effects and establishing the regions on the body where actual pressures differ in magnitude and phase from those predicted by inviscid flow theories

In order to relate the experimentally measured values of the lift with the theory outlined in Ref 1, it is necessary to estimate the net flux of vorticity from the boundary layer into the main stream at each instant of time This problem was generally discussed in Ref 1, and expressions for the transport of vorticity in both steady and unsteady flow were given These equations were justified by experimental observations made in steady flow on slender ellipsoids of revolution which indicated a sharp increase in vorticity in the main stream in the vicinity of the separation line Our present task, therefore, consists in predicting the instantaneous locations of the separation line on our test specimen The difficulties of this problem need hardly be emphasized The tests were conducted at a Reynolds number of 2 7 × 10^6, which implies a turbulent boundary layer Among the many unknowns of the problem, for instance, are the effects of the heaving oscillations of the model on transition and separation Nevertheless, the results of an investigation made by Karlsson[2] on a turbulent boundary layer subjected to a periodic oscillation of the external flow indicate that predictions based on a laminar boundary layer will provide a good estimate of the phase lags involved, although not of the magnitude of the lift Karlsson's work showed that, at the frequency at which our model was oscillated, the vorticity in the boundary layer is unable to redistribute itself instantaneously, so that the flow in the boundary layer is composed of a steady part corresponding to the mean external flow and a time-varying

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